

Angular Momentum and Discrete Symmetries in the Spinor Bethe-Salpeter Equation. II

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The classification of states in the fermion-fermion Bethe-Salpeter equation is discussed. They are related to the states in the fermion-antifermion case discussed before. This enables us to obtain restrictions on the interaction in quark models. Some models used in the literature are thereby found to lead to unphysical predictions.

1. Introduction

The numerical solution of the Bethe-Salpeter (BS) equation in realistic cases is beset by numerous difficulties. Besides the problems related to singular interactions which arise from renormalizable Lagrangians, it is the large number of coupled equations in the spinor-spinor case which renders a numerical solution a difficult task. The importance of the spinor-spinor BS equation must not be stressed here, as physical applications we mention only positronium, various quark models, and nucleon-nucleon scattering.

In order to reduce the number of coupled equations as far as possible one has to investigate the symmetry properties of the equation very carefully. Especially we are interested in symmetries arising from the Poincaré group. These symmetry properties have been exploited in a previous paper¹ (referred to as I in the following) for the fermion-antifermion case. Here we extend this investigation to the fermion-fermion problem.

This case has been studied before in the helicity formalism^{2,3} which is suited for the scattering problem. But we feel that our approach which is guided by the analogy to the non-relativistic angular momentum reduction has its own merits. It will allow for a detailed comparison of the fermion-fermion and fermion-antifermion problem. In respect to the quark model we will demonstrate that a simultaneous treatment of the diquark and quark-antiquark problem may reduce the freedom in choosing the interaction.

2. Fundamentals

We are interested in the symmetry properties under Lorentz transformations of the fermion-fermion BS-amplitude

$$\tau_{\alpha\beta}(x_1, x_2) = \langle 0 | T \psi_\alpha(x_1) \psi_\beta(x_2) | a \rangle. \quad (2.1)$$

The state $|a\rangle$ is assumed to be characterized by momentum P_μ , angular momentum j , spinprojection j_3 , and parity η_P . Furthermore, we have the exchange properties

$$\tau_{\alpha\beta}(x_1, x_2) = -\tau_{\beta\alpha}(x_2, x_1) \quad (2.2)$$

since we are dealing with identical fermions. For simplicity, we deal with fermions without internal symmetries. The incorporation of isospin or SU(3) is straightforward and leads to no problems.

From the quantum numbers of the state $|a\rangle$ we construct eigenvalue equations for the BS amplitude $\tau(x_1, x_2)$. The procedure is completely analogous to the fermion-antifermion case. So we give only the results. For the momentum, we find:

$$\left(-i \frac{\partial}{\partial x_1^\mu} - i \frac{\partial}{\partial x_2^\mu} \right) \tau_{\alpha\beta}(x_1, x_2) = P_\mu \tau_{\alpha\beta}(x_1, x_2) \quad (2.3)$$

which is identical to the fermion-antifermion case. More interesting is the angular momentum. The action of the generator $M_{\mu\nu}$ on the state $|a\rangle$ may be represented as follows:

$$\begin{aligned} \langle 0 | T \{ \psi_\alpha(x_1) \psi_\beta(x_2) \} M_{\mu\nu} | a \rangle \\ = -i \left(x_{1\mu} \frac{\partial}{\partial x_1^\nu} - x_{1\nu} \frac{\partial}{\partial x_1^\mu} + x_{2\mu} \frac{\partial}{\partial x_2^\nu} - x_{2\nu} \frac{\partial}{\partial x_2^\mu} \right) \\ \cdot \tau_{\alpha\beta}(x_1, x_2) \\ + \frac{1}{2} (\Sigma_{\mu\nu})_{\alpha\gamma} \tau_{\gamma\beta}(x_1, x_2) + \frac{1}{2} (\Sigma_{\mu\nu})_{\beta\gamma} \tau_{\alpha\gamma}(x_1, x_2) \end{aligned} \quad (2.4)$$

with

$$\Sigma_{\mu\nu} = \frac{i}{4} (\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu).$$

(2.4) is different from the corresponding fermion-antifermion equation in the spin-dependent part. In a shorthand notation, it now reads $\Sigma_{\mu\nu} \tau + \tau \Sigma_{\mu\nu}^T$ compared to $\Sigma_{\mu\nu} \tau - \tau \Sigma_{\mu\nu}$ before. This difference



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may be remedied if we consider instead of T the quantity

$$\tau'_{\alpha\beta}(x_1, x_2) = \tau_{\alpha\gamma}(x_1, x_2) C_{\gamma\beta}^{-1} \quad (2.5)$$

where $C_{\gamma\beta}$ is the charge conjugation matrix. Since $C \Sigma_{\mu\nu}^T C^{-1} = -\Sigma_{\mu\nu}$ we now have instead of (2.4):

$$\begin{aligned} & \langle 0 | T \{ \psi_\alpha(x_1) \psi_\gamma(x_2) \} M_{\mu\nu} | a \rangle C_{\gamma\beta}^{-1} \\ &= -i \left(x_{1\mu} \frac{\partial}{\partial x_{1\nu}} - x_{1\nu} \frac{\partial}{\partial x_{1\mu}} + x_{2\mu} \frac{\partial}{\partial x_{2\nu}} - x_{2\nu} \frac{\partial}{\partial x_{2\mu}} \right) \\ & \quad \tau'_{\alpha\beta}(x_1, x_2) \quad (2.6) \\ &+ \frac{1}{2} (\Sigma_{\mu\nu})_{\alpha\gamma} \tau'_{\gamma\beta}(x_1, x_2) - \frac{1}{2} (\Sigma_{\mu\nu})_{\gamma\beta} \tau'_{\alpha\gamma}(x_1, x_2). \end{aligned}$$

(2.6) is identical to the fermion-antifermion case. Therefore we will use $\tau'_{\alpha\beta}(x_1, x_2)$ in the rest of this paper. (2.3) holds equally for τ' since it does not involve the discrete indices.

The question of parity is somewhat intricate. If we define $\bar{x} = i_s x = (x_0, -\mathbf{x})$ we have the transformation law for the field operator

$$U(i_s) \psi(x) U^{-1}(i_s) = \eta \gamma_0 \psi(\bar{x}) \quad (2.7)$$

with a phase factor η . Since two successive reflections are identical to a rotation of 2π or a restoration of the beginning we have $\eta^2 = \pm 1$ or $\eta = \pm 1, \pm i$. Since for the adjoint field operator

$$U(i_s) \bar{\psi}(x) U^{-1}(i_s) = \eta^* \bar{\psi}(\bar{x}) \gamma_0, \quad (2.8)$$

we get $\eta\eta^* = 1$ in the fermion-antifermion case. So the actual value of η does not matter there. But in the fermion-fermion case we have a change in sign depending on η . It will, however, be of no importance since parity may be defined consistently only on parts of the complete Hilbert space with constant fermion number⁴. So we may safely put $\eta = 1$ thereby fixing the relation of parity in different subspaces. The eigenvalue equation for τ' is now derived readily. We write it down immediately for the relative wave function $\varphi'(x)$:

$$\gamma_0 \varphi'(\bar{x}) \gamma_0 = -\eta_p \varphi'(x) \quad (2.9)$$

with a change in sign compared to I. This is the expression of the fact that antiparticles have opposite intrinsic parity compared to particles.

Finally, we have to transcribe the Pauli principle (2.2) to φ' . We find

$$C \varphi'^T(-x) C^{-1} = \varphi'(x). \quad (2.10)$$

This is identical to the charge conjugation equation in I with $\eta_c = +1$. We might get a different sign on the righthand side of (2.10) by the introduction of internal symmetries.

3. Classification of States

Since the eigenvalue equations derived in the preceding section are identical to those given in I, we can immediately write down the complete classification of states in terms of the quantum numbers. τ' is first divided into the centre of mass motion and a relative wave function according to (2.3):

$$\tau'_{\alpha\beta}(x_1, x_2) = \varphi_{\alpha\beta}(x_1 - x_2) \exp \{ i P(x_1 + x_2)/2 \}. \quad (3.1)$$

$\varphi_{\alpha\beta}$ is expanded in terms of the elements of the Dirac algebra:

$$\begin{aligned} \varphi(z) = & S(z) + \gamma_5 P(z) + \gamma_0 V_4(z) + \gamma_0 \gamma_5 A_4(z) \\ & + \sum_{i=1}^3 \{ \gamma_i V_i(z) + \gamma_i \gamma_5 A_i(z) + \sigma_{0i} U_i(z) \\ & + \varepsilon_{ijk} \sigma_{jk} T_i(z) \}. \end{aligned} \quad (3.2)$$

Finally, the different amplitudes S, V_i etc. are then expanded in terms of spherical harmonics and vector spherical harmonics, respectively:

$$\begin{aligned} S(z) = & Y_{j\bar{j}}(\Omega_z) s_{j\bar{j}}(z_0, r), \\ V_i(z) = & Y_{(jj-1\bar{j})i}(\Omega_z) v_{j\bar{j}}^{(-)}(z_0, r) \\ & + Y_{(jj\bar{j})i}(\Omega_z) v_{j\bar{j}}^{(0)}(z_0, r) \\ & + Y_{(jj+1\bar{j})i}(\Omega_z) v_{j\bar{j}}^{(+)}(z_0, r), \end{aligned} \quad (3.3)$$

Table 1. The different states of the system and the amplitudes belonging to them. (The subscripts "e" and "o" refer to the even or odd behaviour of the amplitudes under reversal of relative time.)

Parity $(-1)^{j+1}$				Parity $(-1)^j$			
j even Triplet $j=l \pm 1$		j odd Triplet $j=l \pm 1$		j even Singlet $j=l$		j odd Triplet $j=l$	
$j\eta_p$	States	$j\eta_p$	States	$j\eta_p$	States	$j\eta_p$	States
0 ⁻	³ P ₀	1 ⁺	³ S ₁ + ³ D ₁	0 ⁺	¹ S ₀	1 ⁻	³ P ₁
2 ⁻	³ P ₂ + ³ F ₂	3 ⁺	³ D ₃ + ³ G ₃	2 ⁺	¹ D ₂	3 ⁻	³ F ₃
Amplitudes		Amplitudes		Amplitudes		Amplitudes	
s ₀ , v _e ⁽⁴⁾ , a ₀ ⁽⁰⁾ , t _e ⁽⁰⁾ , v ₀ ⁽⁺⁾ , v ₀ ⁽⁻⁾ , u ₀ ⁽⁺⁾ , u ₀ ⁽⁻⁾		s _e , v ₀ ⁽⁴⁾ , a _e ⁽⁰⁾ , t ₀ ⁽⁰⁾ , v _e ⁽⁺⁾ , v _e ⁽⁻⁾ , u _e ⁽⁺⁾ , u _e ⁽⁻⁾		p ₀ , a _e ⁽⁴⁾ , v ₀ ⁽⁰⁾ , u ₀ ⁽⁰⁾ , a ₀ ⁽⁺⁾ , a ₀ ⁽⁻⁾ , t _e ⁽⁺⁾ , t _e ⁽⁻⁾		p _e , a ₀ ⁽⁴⁾ , v _e ⁽⁰⁾ , u _e ⁽⁰⁾ , a _e ⁽⁺⁾ , a _e ⁽⁻⁾ , t ₀ ⁽⁺⁾ , t ₀ ⁽⁻⁾	

and so on. For brevity, we will omit the subscript $j j_3$ in the following. The coupling of the various amplitudes and the quantum numbers belonging to them are shown in Table 1. It is obtained from the corresponding table in I by some changes in sign. The only important difference is that now we have a selection of the allowed combinations according to the angular momentum. Furthermore, we now have physical states also for those amplitudes which correspond to exotics of the second kind in the fermion-antifermion case (amplitudes s_0 etc.).

In the scattering problem, an enormous simplification arises. If the external particles are on the mass-shell the relative time is zero for purely kinematical reasons. Thus all amplitudes with an "o" subscript vanish.

4. Quark Model Considerations

We now turn to the dynamical relation between bound states in fermion-fermion and fermion-antifermion case. For definiteness, we will use the quark model but our considerations apply to all systems as well.

The BS-equation for a quark-antiquark bound state in the ladder approximation reads in momentum space:

$$\begin{aligned} & [m - \gamma^\mu (\tfrac{1}{2} P_\mu + q_\mu)] \tau^{\text{FA}}(q, P) [m + \gamma^\nu (\tfrac{1}{2} P_\nu + q_\nu)] \\ &= \frac{\lambda}{\pi^2 i} \int d^4 q' \{ I_s(q - q') \tau^{\text{FA}}(q', P) \\ &+ I_p(q - q') \gamma_5 \tau^{\text{FA}}(q', P) \gamma_5 \\ &+ I_V^\mu(q - q') \gamma_\mu \tau^{\text{FA}}(q', P) \gamma_\nu \\ &+ I_A^\mu(q - q') \gamma_5 \gamma_\mu \tau^{\text{FA}}(q', P) \gamma_5 \gamma_\nu \}. \end{aligned} \quad (4.1)$$

The corresponding diquark equation is:

$$\begin{aligned} & [m - \gamma^\mu (\tfrac{1}{2} P_\mu + q_\mu)] \tau^{\text{FF}}(q, P) [m + \gamma^\nu (\tfrac{1}{2} P_\nu + q_\nu)] \\ &= \frac{\lambda}{\pi^2 i} \int d^4 q' \{ I_s(q - q') \tau^{\text{FF}}(q', P) \\ &+ I_p(q - q') \gamma_5 \tau^{\text{FF}}(q', P) \gamma_5 \\ &- I_V^\mu(q - q') \gamma_\mu \tau^{\text{FF}}(q', P) \gamma_\nu \\ &+ I_A^\mu(q - q') \gamma_5 \gamma_\mu \tau^{\text{FF}}(q', P) \gamma_5 \gamma_\nu \}. \end{aligned} \quad (4.2)$$

It differs from (4.1) only in the sign of the vector interaction if we derive the BS kernels from an underlying Lagrangian. Of course, there will be no relation at all between the two cases if we use phenomenological interactions. But we believe that even if one adopts such a phenomenological standpoint one should preserve the connection in (4.1) and (4.2) in order to reduce the number of free parameters.

In the absence of vector interactions the two equations (4.1), (4.2) will be identical. Therefore, they will have an identical spectrum of eigenvalues. So degenerate in mass to the π one will find a diquark bound state with $j^{\text{np}} = 0^+$, and the ϱ will be accompanied by a 1^+ diquark. This is in sharp contrast with the experimental situation where not a single diquark state has been found. So we conclude that a vector interaction is absolutely necessary in order to reproduce the experimental situation. Therefore all quark models without a vector interaction should be discarded, e. g. that of Joos et al.⁵

This statement is not spoiled if we consider the normalization of the BS amplitudes. The normalization integral involves essentially only the kinetic term which is identical in both cases.

A reasonable quark model, in our belief, should not only give a reasonable meson mass spectrum, but also explain the absence of bound diquark states without introducing additional parameters. This requirement enforces the presence of a strong vector interaction. But whether it is possible to fulfill remains to be shown by explicit dynamical calculations.

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